

APPROXIMATE DERIVATIONS OF ORDER n

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ABSTRACT. The aim of this paper is to prove characterization theorems for higher order derivations. Among others we prove that the system defining higher order derivations is stable. Further characterization theorems in the spirit of N. G. de Bruijn will also be presented.

1. INTRODUCTION

Throughout this paper \mathbb{N} denotes the set of the positive integers, further \mathbb{Z}, \mathbb{Q} , and \mathbb{R} have the usual meaning.

The aim of this work is to prove characterization theorems on derivations as well as on linear functions. Therefore, firstly we have to recall some definitions and auxiliary results. Unless we indicate in the other way, all the notions follow the monograph Kuczma [6].

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called an *additive* function if,

$$f(x + y) = f(x) + f(y)$$

holds for all $x, y \in \mathbb{R}$.

We say that an additive function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a *derivation* if

$$f(xy) = xf(y) + yf(x)$$

is fulfilled for all $x, y \in \mathbb{R}$.

The additive function $f : \mathbb{R} \rightarrow \mathbb{R}$ is termed to be a *linear function* if f is of the form

$$f(x) = f(1)x \quad (x \in \mathbb{R}).$$

For any function $f : \mathbb{R} \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}$ we define

$$\delta_\alpha f(x) = f(\alpha x) - \alpha f(x) \quad (x \in \mathbb{R}).$$

Clearly, if $f : \mathbb{R} \rightarrow \mathbb{R}$ is an additive function then

$$\delta_\alpha f(x) = 0 \quad (\alpha, x \in \mathbb{R})$$

or

$$\delta_\alpha f(1) = 0 \quad (\alpha \in \mathbb{R})$$

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yield that f is linear.

Let $n \in \mathbb{N}$ be fixed. Following the work [13] of J. Unger and L. Reich, an additive function $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to be a *derivation of order n* if,

$$f(1) = 0 \quad \text{and} \quad \delta_{\alpha_1} \circ \cdots \circ \delta_{\alpha_{n+1}} f(x) = 0$$

is fulfilled for any $x, \alpha_1, \dots, \alpha_{n+1} \in \mathbb{R}$.

From this notion immediately follows that first order derivations are just real derivations. If we drop the assumption $f(1) = 0$, we get that an additive function $f: \mathbb{R} \rightarrow \mathbb{R}$ fulfills

$$\delta_\alpha \delta_\beta f(x) = 0 \quad (\alpha, \beta, x \in \mathbb{R})$$

if and only if

$$f(x) = d(x) + f(1)x,$$

where d is a real derivation.

Furthermore, we also remark that there exists derivations of second order that are not first order derivations. Namely, let $d: \mathbb{R} \rightarrow \mathbb{R}$ be a nontrivial derivation and consider the function $d \circ d$.

In the remainder part of this section, we will summarize those results that will be utilized in connection with multiadditive and polynomial functions, respectively. For further details we refer to Székelyhidi [10, 11, 12].

Let G, H be abelian groups, let $h \in G$ be arbitrary and consider a function $f: G \rightarrow H$. The *difference operator* Δ_h with the span h of the function f is defined by

$$\Delta_h f(x) = f(x + h) - f(x) \quad (x \in G).$$

The iterates Δ_h^n of Δ_h , $n = 0, 1, \dots$ are defined by the recurrence

$$\Delta_h^0 f = f, \quad \Delta_h^{n+1} f = \Delta_h (\Delta_h^n f) \quad (n = 0, 1, \dots)$$

Furthermore, the superposition of several difference operators will be denoted concisely

$$\Delta_{h_1 \dots h_n} f = \Delta_{h_1} \dots \Delta_{h_n} f,$$

where $n \in \mathbb{N}$ and $h_1, \dots, h_n \in G$.

Let $n \in \mathbb{N}$ and G, H be abelian groups. A function $F: G^n \rightarrow H$ is called *n -additive* if, for every $i \in \{1, 2, \dots, n\}$ and for every $x_1, \dots, x_n, y_i \in G$,

$$\begin{aligned} & F(x_1, \dots, x_{i-1}, x_i + y_i, x_{i+1}, \dots, x_n) \\ &= F(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) + F(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n), \end{aligned}$$

i.e., F is additive in each of its variables $x_i \in G$, $i = 1, \dots, n$. For the sake of brevity we use the notation $G^0 = G$ and we call constant functions from G to H 0-additive functions. Let $F: G^n \rightarrow H$ be an arbitrary function. By the *diagonalization (or trace)* of F we understand the function $f: G \rightarrow H$ arising from F by putting all the variables (from G) equal:

$$f(x) = F(x, \dots, x) \quad (x \in G).$$

It can be proved by induction that for any symmetric, n -additive function $F: G^n \rightarrow H$ the equality

$$\Delta_{y_1, \dots, y_k} f(x) = \begin{cases} n! F(y_1, \dots, y_n) & \text{for } k = n \\ 0 & \text{for } k > n \end{cases}$$

holds, whenever $x, y_1, \dots, y_n \in G$, where $f: G \rightarrow H$ denotes the trace of the function F . This means that a symmetric, n -additive function is uniquely determined by its trace.

The function $f: G \rightarrow H$ is called a *polynomial function* of degree at most n , where n is a nonnegative integer, if

$$\Delta_{y_1, \dots, y_{n+1}} f(x) = 0$$

is satisfied for all $x, y_1, \dots, y_{n+1} \in G$.

Theorem 1.1 (Kuczma [6], Székelyhidi [12]). *The function $p: G \rightarrow H$ is a polynomial of degree at most n if and only if there exist symmetric, k -additive functions $F_k: G^k \rightarrow H$, $k = 0, 1, \dots, n$ such that*

$$p(x) = \sum_{k=0}^n f_k(x) \quad (x \in G),$$

where f_k denotes the trace of the function F_k , $k = 0, 1, \dots, n$. Furthermore, this expression for the function p is unique in the sense that the functions F_k , which are not identically zero, are uniquely determined.

Henceforth we will say that the function in question is *locally regular* on its domain, if at least one of the following statements are fulfilled.

- (i) bounded on a measurable set of positive measure;
- (ii) continuous at a point;
- (iii) there exists a set of positive Lebesgue measure so that the restriction of the function in question is measurable in the sense of Lebesgue.

Lemma 1.2. *Let $n \in \mathbb{N}$ be fixed and $p: \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial of degree at most n . If p is locally regular then p is a real polynomial, i.e.,*

$$p(x) = a_n x^n + \dots + a_1 x + a_0 \quad (x \in \mathbb{R})$$

with certain real constants a_0, a_1, \dots, a_n .

Corollary 1.3. *Let $n \in \mathbb{N}$ be arbitrarily fixed and $F: \mathbb{R} \rightarrow \mathbb{R}$ be a symmetric, n -additive function. Assume further, that the trace of F is locally regular, then there exists a constant $c \in \mathbb{R}$ such that*

$$F(x_1, \dots, x_n) = c x_1 \cdots x_n \quad (x \in \mathbb{R}).$$

2. RESULTS

Preparatory statements. We begin with the following characterization of n^{th} order derivations.

Theorem 2.1. *Let $n \in \mathbb{N}$ be fixed and $f: \mathbb{R} \rightarrow \mathbb{R}$ be an additive function and assume that the mapping*

$$\mathbb{R} \ni \alpha \mapsto \delta_\alpha^{n+1} f(1)$$

is locally regular. Then and only then there exists an n^{th} order derivation $d: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) = d(x) + f(1) \cdot x \quad (x \in \mathbb{R}).$$

Proof. Define the function $F: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ through

$$F(\alpha_1, \dots, \alpha_{n+1}) = \delta_{\alpha_1} \circ \dots \circ \delta_{\alpha_{n+1}} f(1) \quad (\alpha_1, \dots, \alpha_{n+1} \in \mathbb{R}).$$

Due to the additivity of f , the mapping F is a symmetric, $(n+1)$ -additive function. Furthermore, its trace, that is,

$$F(\alpha, \dots, \alpha) = \delta_\alpha^{n+1} f(1) \quad (\alpha \in \mathbb{R})$$

is locally regular. Thus there exists a constant $c \in \mathbb{R}$ so that

$$F(\alpha_1, \dots, \alpha_{n+1}) = c\alpha_1 \cdots \alpha_{n+1} \quad (\alpha_1, \dots, \alpha_{n+1} \in \mathbb{R}).$$

Put $\alpha_i = 1$ for all $i = 1, \dots, n+1$ to get

$$0 = F(1, \dots, 1) = c.$$

This means that the mapping F is identically zero, in other words, we have

$$\delta_{\alpha_1} \circ \dots \circ \delta_{\alpha_{n+1}} f(1) = 0$$

for all $\alpha_1, \dots, \alpha_{n+1} \in \mathbb{R}$. This however yields that the function $f - f(1)\text{id}$ is an n^{th} order derivation. \square

In view of the previous result, the following corollary can be verified easily.

Corollary 2.2. *Let $n \in \mathbb{N}$ be fixed and $f: \mathbb{R} \rightarrow \mathbb{R}$ be an additive function and assume that the mapping*

$$\mathbb{R}^{n+1} \ni (\alpha_1, \dots, \alpha_{n+1}) \mapsto \delta_{\alpha_1} \circ \delta_{\alpha_2} \circ \dots \circ \delta_{\alpha_{n+1}} f(1)$$

is locally regular. Then and only then there exists an n^{th} order derivation $d: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) = d(x) + f(1) \cdot x \quad (x \in \mathbb{R}).$$

Proof. Similarly, as in the proof of the previous theorem, let

$$F(\alpha_1, \dots, \alpha_{n+1}) = \delta_{\alpha_1} \circ \dots \circ \delta_{\alpha_{n+1}} f(1) \quad (\alpha_1, \dots, \alpha_{n+1} \in \mathbb{R}).$$

Because of the additivity of the function f , the mapping F is a locally regular, symmetric and n -additive function. As in the previous proof, from this we conclude that $f - f(1)\text{id}$ is an n^{th} order derivation. \square

Hyers type results. In this section we will investigate whether the system defining higher order derivations is stable in the sense of Hyers. To answer this problem affirmatively, we will use the following result of D. H. Hyers, see [5].

Theorem 2.3 (Hyers). *Let $\varepsilon \geq 0$, X, Y be Banach spaces and $f : X \rightarrow Y$ be a function. Suppose that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

holds for all $x, y \in X$. Then, for all $x \in X$, the limit

$$a(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$$

does exist, the function $a : X \rightarrow \mathbb{R}$ is additive on X , i.e.,

$$a(x+y) = a(x) + a(y)$$

holds for all $x, y \in X$, furthermore,

$$\|f(x) - a(x)\| \leq \varepsilon$$

is fulfilled for arbitrary $x \in X$. Additionally, the function $a : X \rightarrow Y$ is uniquely determined by the above formula.

The above theorem briefly expresses the following. Assume that X, Y are Banach spaces and the function $f : X \rightarrow Y$ satisfies the additive Cauchy equation only 'approximatively'. Then there exists a unique additive function $a : X \rightarrow Y$ which is 'close' to the function f . Since 1941 this result has been extended and generalized in a several ways. Furthermore, such a problem can obviously be raised concerning not only the Cauchy equation but also in connection with other equations and system of equations, as well.

In [1] R. Badora investigated the above stability problem for derivations. Furthermore, in [9] J. Schwaiger proved a stability type result for higher order derivations. Concerning stability of real derivations as well as linear functions, we refer to [2, 4].

Our aim is to generalize the results archived by R. Badora and J. Schwaiger and at the same time we will show that with a different idea the proofs can be shortened significantly.

Concerning higher order derivations, we will prove the following.

Theorem 2.4. *Let $n \in \mathbb{N}$ and $\varepsilon \geq 0$. Assume that for the function $f : \mathbb{R} \rightarrow \mathbb{R}$*

$$|f(x+y) - f(x) - f(y)| < \varepsilon \quad (x \in \mathbb{R})$$

and that the mapping

$$\mathbb{R} \times \mathbb{R} \ni (\alpha, x) \mapsto \delta_\alpha^{n+1} f(x)$$

is locally bounded. Then there exist an n^{th} order derivation $d : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$|f(x) - [d(x) + f(1) \cdot x]| < \varepsilon$$

for all $x \in \mathbb{R}$.

Proof. Since the function $f: \mathbb{R} \rightarrow \mathbb{R}$ is approximatively additive, that is,

$$|f(x+y) - f(x) - f(y)| < \varepsilon \quad (x \in \mathbb{R}),$$

due to the theorem of Hyers we immediately get that there exists an additive function $a: \mathbb{R} \rightarrow \mathbb{R}$ and a bounded function $b: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) = a(x) + b(x) \quad (x \in \mathbb{R})$$

further,

$$|b(x)| < \varepsilon \quad (x \in \mathbb{R}).$$

Additionally, there exists a locally bounded function $B: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$\delta_\alpha^{n+1} f(x) = B(\alpha, x) \quad (\alpha, x \in \mathbb{R}).$$

Therefore,

$$\delta_\alpha^{n+1} a(1) = \delta_\alpha^{n+1} f(1) - \delta_\alpha^{n+1} b(1) = B(\alpha, 1) - \delta_\alpha^{n+1} b(1) \quad (\alpha \in \mathbb{R}).$$

Let us observe that in the previous expression, the right hand side is a locally bounded function. Thus by Theorem 2.1. there exists an n^{th} order derivation $d: \mathbb{R} \rightarrow \mathbb{R}$ so that

$$a(x) = d(x) + a(1)x \quad (x \in \mathbb{R}),$$

yielding that

$$|f(x) - [d(x) + \lambda x]| < \varepsilon \quad (x \in \mathbb{R}),$$

where $\lambda = a(1)$. □

de Bruijn type results. Finally, we will close this section with a result that concerns also characterization theorem on derivations in the spirit of G. N. de Bruijn.

In [3] he introduced the following notion, for further results we refer to the PhD dissertation of Gy. Maksa, see [8].

The set of real functions $\mathcal{X} \subset \mathbb{R}^{\mathbb{R}}$ is said to have *the difference property* if the following is fulfilled: if for all $h \in \mathbb{R}$ the function $\Delta_h f \in \mathcal{X}$, then there exists an additive function a and a function $\gamma \in \mathcal{X}$ such that

$$f(x) = a(x) + \gamma(x) \quad (x \in \mathbb{R}).$$

Subsequently, the following function spaces will be important for us.

Let $\Omega \subset \mathbb{R}$ be an open set and denote

$$\mathcal{C}^k(\Omega) = \{f: \Omega \rightarrow \mathbb{R} \mid f \text{ is } k\text{-times continuously differentiable}\}$$

$$\mathcal{D}^k(\Omega) = \{f: \Omega \rightarrow \mathbb{R} \mid f \text{ is } k\text{-times differentiable}\}$$

$$\mathcal{C}^\omega(\Omega) = \{f: \Omega \rightarrow \mathbb{R} \mid f \text{ is analytic}\}$$

$$\mathcal{R}(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is Riemann integrable on every bounded interval}\}$$

$$\mathcal{AC}(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is absolutely continuous on every bounded interval}\}$$

$\mathcal{BV}(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ has bounded variation on every bounded interval}\}$

Theorem 2.5 (de Bruijn). *All of the linear spaces listed above possess the difference property.*

For the sake of brevity, if $f: \mathbb{R} \rightarrow \mathbb{R}$ is a function and $n \in \mathbb{N}$, then we will write

$$(\mathbf{D}f)(\alpha) = \delta_\alpha^{n+1} f(1) \quad (\alpha \in \mathbb{R}).$$

In contrast to the statement of the previous subsection, here we will assume not that the Cauchy difference is bounded, but we suppose that all the first order differences of the function in question are regular.

Theorem 2.6. *Let $n \in \mathbb{N}$ be arbitrarily fixed and $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function. Assume furthermore at least one of the following statements are fulfilled.*

- (i) *there exists $k \in \mathbb{N} \cup \{0\}$ such that for all $h \in \mathbb{R}$ $\Delta_h f \in \mathcal{C}^k(\mathbb{R})$ and $\mathbf{D}f$ continuous at least one point.*
 - (ii) *there exists $k \in \mathbb{N}$ such that for all $h \in \mathbb{R}$ $\Delta_h f \in \mathcal{D}^k(\mathbb{R})$ and $\mathbf{D}f$ continuous at least one point.*
 - (iii) *for all $h \in \mathbb{R}$ $\Delta_h f \in \mathcal{C}^\omega(\mathbb{R})$ and $\mathbf{D}f$ continuous at least one point.*
 - (iv) *for all $h \in \mathbb{R}$ $\Delta_h f \in \mathcal{R}(\mathbb{R})$ and $\mathbf{D}f$ fulfills regularity condition (i) or (iii).*
 - (v) *for all $h \in \mathbb{R}$ $\Delta_h f(x) \in \mathcal{AC}(\mathbb{R})$ and $\mathbf{D}f$ fulfills regularity condition (i) or (iii).*
 - (vi) *for all $h \in \mathbb{R}$ $\Delta_h f(x) \in \mathcal{BV}(\mathbb{R})$ and $\mathbf{D}f$ fulfills regularity condition (i) or (iii).*
- Then and only then there exists an n^{th} order derivation $d: \mathbb{R} \rightarrow \mathbb{R}$ and a function $\gamma: \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$f(x) = d(x) + \lambda x + \gamma(x)$$

for all $x \in \mathbb{R}$, here the function γ is, according to the above cases k -times continuously differentiable, k -times differentiable, analytic, Riemann integrable, absolutely continuous and of bounded variation, respectively.

Proof. Assume that at least one of (i), (ii) and (iii) is fulfilled and let \mathfrak{A} denote any of the spaces appearing in this statements. Since \mathfrak{A} has the difference property we get that there exists an additive function $a: \mathbb{R} \rightarrow \mathbb{R}$ and $\gamma \in \mathfrak{A}$ such that

$$f(x) = a(x) + \gamma(x) \quad (x \in \mathbb{R}).$$

On the other hand, the mapping $\mathbf{D}f$ is continuous at least one point. Thus there exists a function $R: \mathbb{R} \rightarrow \mathbb{R}$ which is continuous at least one point so that

$$(\mathbf{D}f)(\alpha) = R(\alpha) \quad (\alpha \in \mathbb{R}).$$

This implies that for the additive function a ,

$$\delta_\alpha^{n+1} a(1) = R(\alpha) - \delta_\alpha^{n+1} \gamma(1) \quad (\alpha \in \mathbb{R}).$$

Since $\gamma \in \mathfrak{A}$, the function γ is continuous everywhere. Therefore the right hand side in the previous representation is continuous at least one point. Thus by Theorem 2.1, there exists an n^{th} order derivation $d: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$a(x) = d(x) + a(1)x \quad (x \in \mathbb{R}),$$

from which we get that

$$f(x) = d(x) + \lambda x + \gamma(x) \quad (x \in \mathbb{R}),$$

where $\lambda = a(1)$.

Now assume that at least one of (iv), (v) and (vi) is fulfilled and let

$$\mathfrak{I} \in \{\mathcal{R}(\mathbb{R}), \mathcal{AC}(\mathbb{R}), \mathcal{BV}(\mathbb{R})\}.$$

Since the linear space \mathfrak{I} has the finite difference property, we immediately get that there exists an additive function $a: \mathbb{R} \rightarrow \mathbb{R}$ and $\gamma \in \mathfrak{I}$ such that

$$f(x) = a(x) + \gamma(x) \quad (x \in \mathbb{R}).$$

On the other hand there exists a function $R: \mathbb{R} \rightarrow \mathbb{R}$ fulfilling regularity assumption (i) or (ii) such that

$$\delta_\alpha^{n+1} f(1) = R(\alpha) \quad (\alpha \in \mathbb{R}).$$

For the additive function a this yields that

$$\delta_\alpha^{n+1} a(1) = R(\alpha) - \delta_\alpha^{n+1} \gamma(1) \quad (\alpha \in \mathbb{R}).$$

Since any element of the linear space \mathfrak{I} fulfills (i) as well as (ii) the right hand side of the above representation satisfies regularity assumption (i) or (iii). Applying Theorem 2.1, there exists an n^{th} order derivation $d: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$a(x) = d(x) + a(1)x \quad (x \in \mathbb{R}).$$

From which the statement follows. \square

3. A REMARK

Due to a result of M. Laczkovich (see [7]), the set of all measurable real functions does not have the difference property. More precisely, the following statement holds.

Theorem 3.1. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be function such that for all $h \in \mathbb{R}$ the function $\Delta_h f$ is measurable. Then there exist an additive function $a: \mathbb{R} \rightarrow \mathbb{R}$, a measurable function $\gamma: \mathbb{R} \rightarrow \mathbb{R}$ and a function $\sigma: \mathbb{R} \rightarrow \mathbb{R}$ so that for almost all $x, h \in \mathbb{R}$,*

$$\sigma(x + h) = \sigma(x)$$

and

$$f(x) = a(x) + \sigma(x) + \gamma(x) \quad (x \in \mathbb{R}).$$

Assuming the Continuum Hypothesis, there exists a *nonmeasurable* function $\sigma: \mathbb{R} \rightarrow \mathbb{R}$ with the above properties. This shows that the set of measurable functions does not have the difference property.

Nevertheless, if we assume additionally that the mapping $\mathbf{D}f$ is also measurable then it can happen that the function σ in the theorem of Laczkovich should be measurable. Thus we can formulate the following problem.

Open problem 3.2. Let $n \in \mathbb{N}$ be fixed and $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function. Assume that for all $h \in \mathbb{R}$ the function Δ_h is measurable, additionally the mapping

$$\mathbb{R} \ni \alpha \longmapsto \delta_\alpha^{n+1} f(1)$$

is measurable, as well. Is it true that there exists a derivation $d: \mathbb{R} \rightarrow \mathbb{R}$ of order n and a measurable function $\gamma: \mathbb{R} \rightarrow \mathbb{R}$ so that

$$f(x) = d(x) + \gamma(x)$$

for all $x \in \mathbb{R}$.

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